PROJECTED WRITTEN NOTES FROM THE M325K LECTURE ON THURSDAY, FEBRUARY 29, 2024,
on SECTION 6.1 –
AN INTRODUCTION TO SET THEORY

CLASS #14

SET THEORY

Conceptual DEFINITION:
A set is a collection of "objects" called elements or member of the set.

Ex: let $D = \{a, b, c\}$ ← "The Set Containing..." ← [The List or Roster NOTATION]

Let $B = \{2, \{1\}, \{8,9,6,5\}\}$.

$2 \in B$, $1 \notin B$, $\{1\} \in B$. $1 \in \{1\}$.

SET BUILDER NOTATION: $\{(\text{all})\ x \in \text{Domain} \mid P(x) \text{ is True}\}$ ↑ "such that"

Ex: let $C = \{x \in \mathbb{Z} \mid 4 \le x \le 8\}$

The List NOTATION for C: $C = \{4, 5, 6, 7, 8\}$

$C = \{4, 4, 7, 6, 6, 6, 5, 8\}$

In any Discussion of Sets, some largest set $\mathcal{U}$, called the Universal set or the Universe of Discussion (explicitly specified or implicitly understood) is a set that contains all of the elements of all of the sets under discussion.

Ex: Let $U = \{1, 2, 3, \ldots, 20\}$

Let $H = \{1, 2, 3\}$, $J = \{2, 3, 5\}$,

$K = \{1, 3\}$, $L = \{3, 2, 1\}$.

See the definitions of $A \cup B$, $A \cap B$, A^c, $A - B$ in the handout "In the Book Definitions (II)"

VENN DIAGRAMS

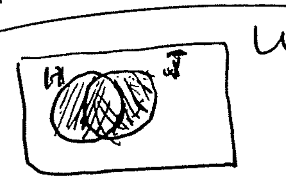
$H \cup J = \{1, 2, 3, 5\}$,

$H \cap J = \{2, 3\}$

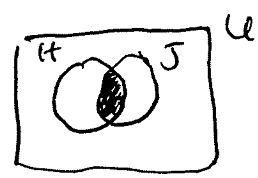
$H^c = \{4, 5, 6, \ldots, 20\}$

$J - H = \{5\} = J \cap H^c$.

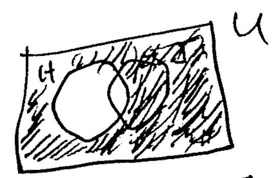
$H \cup J$



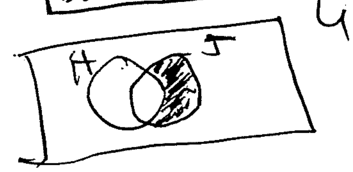
$H \cap J$



H^c



$J - H$



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In-the-Book Definitions (II)

Set U is the Universal Set. Any sets discussed are subsets of the Universal Set U.

Set A is *non-empty* ($A \neq \varnothing$) $\Leftrightarrow$ There exists an element $x \in U$ such that $x \in A$.

The Union of set A and set B is the set $A \cup B$ where:

$$A \cup B = \{x \in U \mid x \in A \text{ OR } x \in B\}.$$

The *procedural* definition of $A \cup B$ which is useful for writing proofs involving $A \cup B$ is:

$$x \in A \cup B \Leftrightarrow x \in A \text{ OR } x \in B.$$

Whenever a set X is defined in the form

$$X = \{x \in U \mid \text{Predicate P is true about } x\},$$

the *procedural* definition of X is: $x \in X \Leftrightarrow$ Predicate P is true about x.

The Intersection ($\cap$) and Difference ($-$) of sets A and B, and the Complement (A^c) of set A, are defined as follows:

$A \cap B = \{x \in U \mid x \in A \text{ AND } x \in B\}$; $\quad A^c = \{x \in U \mid x \notin A\}$;

$A - B = \{x \in U \mid x \in A \text{ AND } x \notin B\}$

For all $x \in A$, $x \in B$.

Set A *is a subset of* Set B ($A \subseteq B$) $\Leftrightarrow$ For all elements $x \in U$, if $x \in A$, then $x \in B$.

A collection $\{A_1, A_2, \dots A_n\}$ of non-empty subsets of A is a Partition of set A

$\Leftrightarrow$ 1) $A = A_1 \cup A_2 \cup \dots \cup A_n$ and 2) when $i \neq j$, $A_i \cap A_j = \varnothing$.

Given a set A, the Power Set of A, denoted $\mathcal{P}(A)$ is the set of all subsets of A.
When A is finite with n elements, then $\mathcal{P}(A)$ has 2^n elements.

Set Equality: Set A = Set B $\Leftrightarrow$ $A \subseteq B$ and $B \subseteq A$.

A (Binary) Relation R from set A to Set B is any subset of the Cartesian Product $A \times B$.

Given such a relation R, for all $a \in A$ and $b \in B$, $a R b \Leftrightarrow (a, b) \in R$.

The Inverse Relation R^{-1} is the subset of $B \times A$ such that
for all $b \in B$ and $a \in A$, $(b, a) \in R^{-1} \Leftrightarrow (a, b) \in R$; thus, $b R^{-1} a \Leftrightarrow a R b$.

A Relation R on A is a relation R from A to A, that is, from A to B with B = A.

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Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \Leftrightarrow x \in X$ or $x \in Y$
2. $x \in X \cap Y \Leftrightarrow x \in X$ and $x \in Y$
3. $x \in X - Y \Leftrightarrow x \in X$ and $x \notin Y$
4. $x \in X^c \Leftrightarrow x \notin X$
5. $(x, y) \in X \times Y \Leftrightarrow x \in X$ and $y \in Y$

Example 6.2.1 Proof of a Subset Relation

Prove Theorem 6.2.1(1)(a): For all sets A and B, $A \cap B \subseteq A$.

Solution We start by giving a proof of the statement and then explain how you can obtain such a proof yourself.

Proof:

Suppose A and B are any sets and suppose x is any element of $A \cap B$. Then $x \in A$ and $x \in B$ by definition of intersection. In particular, $x \in A$. Thus $A \cap B \subseteq A$.

The underlying structure of this proof is not difficult, but it is more complicated than the brevity of the proof suggests. The first important thing to realize is that the statement to be proved is universal (it says that for *all* sets A and B, $A \cap B \subseteq A$). The proof, therefore, has the following outline:

Starting Point: Suppose A and B are any (particular but arbitrarily chosen) sets.

To Show: $A \cap B \subseteq A$

Now to prove that $A \cap B \subseteq A$, you must show that

$$\forall x, \text{ if } x \in A \cap B \text{ then } x \in A.$$

But this statement also is universal. So to prove it, you

suppose x is an element in $A \cap B$

and then you

show that x is in A.

Filling in the gap between the "suppose" and the "show" is easy if you use the procedural version of the definition of intersection: To say that x is in $A \cap B$ means that

x is in A and x is in B.

This allows you to complete the proof by deducing that, in particular,

x is in A,

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For the discussion about proving
a subset Relationship between sets
(eg. "$A \subseteq B$"),

See the handout
"A Proof Design for Proofs of "$A \subseteq B$"

Recall that $H = \{1, 2, 3\}$, $J = \{2, 3, 5\}$
$K = \{1, 3\}$, $L = \{3, 2, 1\}$

$L \subseteq L$.

$K \subseteq L$ because "For all $x \in K$, $x \in L$"
$L \nsubseteq K$ because "There exists the element 2 such that $2 \in L$ and $2 \notin K$."

There is a special set that contains no elements,
the Empty Set or also the NULL SET
and is denoted ϕ.

$\phi = \{\ \}$.

$\phi \subseteq L$, $\phi \subseteq X$ for every set X,

A Proof Design for Proofs of "$A \subseteq B$"

Let A and B be sets.

The definition of "$A \subseteq B$" has two equivalent forms:

Def'n 1:

$A \subseteq B \Leftrightarrow \forall x \in U$, if $x \in A$, then $x \in B$.

Def'n 2:

$A \subseteq B \Leftrightarrow \forall x \in A,\ x \in B$.

These are equivalent because the actual domain of x in Def'n 2 is the same set A as the effective domain of x in Def'n 1.

To Prove: $A \subseteq B$

Proof: Let $x \in A$ be given.

[N.T.S.: $x \in B$]

⋮

$\therefore x \in B$.

$\therefore A \subseteq B$, by Direct Proof.

QED

EXAMPLE:

TO PROVE: For all sets A and B,

$(A \cap B) \subseteq A$.

Proof.

Let A and B be any sets.

[NTS: $(A \cap B) \subseteq A$.]

Let $x \in A \cap B$ be given.

$\therefore x \in A$ AND $x \in B$, by def'n of "intersection".

$\therefore x \in A$, by specialization.

[$\therefore \forall x \in A \cap B,\ x \in A$].

$\therefore (A \cap B) \subseteq A$, by Direct Proof.

$\therefore$ For all sets A and B, $(A \cap B) \subseteq A$, by Direct Proof.

QED.

Proof Design

EXAMPLE PROOF

The Def'n of Set Equality:

$$\text{SET } A = \text{SET } B \iff \underline{A \subseteq B \text{ AND } B \subseteq A}$$

$H \subseteq L$ and $L \subseteq H$, $\therefore$ $H = L$ as sets.

Def'n: Given set A, the Power Set $\mathcal{P}(A)$

is $\mathcal{P}(A) = \{\text{all the subsets of } A\}$.

FACT: when A is a finite set with n elements, the $\mathcal{P}(A)$ has 2^n elements.

Ex: $H = \{1, 2, 3\}$ (3 elements, $n = 3$)

$$\mathcal{P}(H) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}, \phi\}$$

$\mathcal{P}(H)$ has $2^3 = 8$ elements.

H has $2^3 = 8$ subsets.

Def'ns: A, B, C are sets. (ordered pairs)

An ordered duple is (x, y) such that $x \in A$ and $y \in A$.

Let $A = \{5, 6, 7\}$. Here are some ordered pairs:

$B = \{10, 20\}$

$C = \{2\}$

$(5, 6)$,
$(6, 5)$,
$(7, 5)$,
$(5, 5)$,
$\vdots$

Ordered triples (x, y, z), such that $x \in A$, $y \in B$ and $z \in C$, are

$(6, 10, 2)$, $(5, 20, 2)$,

$(7, 10, 2)$,

$(6, 20, 3)$ is Not an ordered Triple since $3 \notin C$.

Suppose A_1, A_2, A_3 are three sets:

The set $A_1 \times A_2 \times A_3$ is the <u>Cartesian Product</u> of sets A_1, A_2, and A_3, and

$$A_1 \times A_2 \times A_3 = \left\{ \begin{array}{l} \text{all ordered triples } (x, y, z) \\ \text{such that } x \in A_1,\ y \in A_2,\ z \in A_3 \end{array} \right\}$$

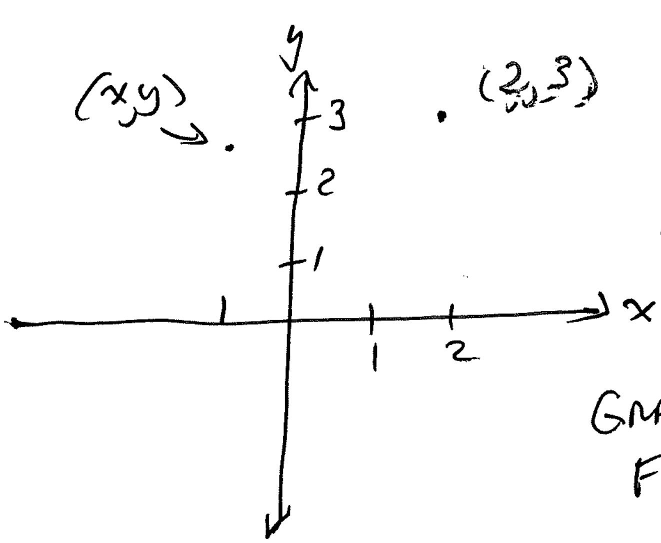
$$A_1 \times A_2 = \left\{ \begin{array}{l} \text{all ordered pair } (x, y) \text{ such that} \\ x \in A_1 \text{ and } y \in A_2 \end{array} \right\}$$

Ex: let $A_1 = \mathbb{Z}^{POS}$, $A_2 = \mathbb{Z}^{POS}$, $A_3 = \mathbb{Z}^{NEG}$.

$$(5, 7, -2) \in \mathbb{Z}^{POS} \times \mathbb{Z}^{POS} \times \mathbb{Z}^{NEG}$$

$$(7, -2, 5) \notin \mathbb{Z}^{POS} \times \mathbb{Z}^{POS} \times \mathbb{Z}^{NEG}$$

$$-2 \notin \mathbb{Z}^{POS},\ 5 \notin \mathbb{Z}^{NEG}$$



$\mathbb{R} \times \mathbb{R}$ is the CARTESIAN PLANE FOR GRAPHING FUNCTIONS.

THE LAST DISCUSSIÓN WAS A PRESENTATION OF THE PROOF ON THE TOP HALF OF page 7 of the Handout "EXAMPLES OF ELEMENTAL PROOFS of INCLUSIÓN in SET THEORY."